

Gravitational Anomaly and Hydrodynamics

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Abstract. We study the anomalous induced current of a vortex in a relativistic fluid via the chiral vortical effect, which is analogous to the anomalous current induced by a magnetic field via the chiral magnetic effect. We perform this analysis at weak and strong coupling. We discuss inequivalent implementations to the chemical potential for an anomalous symmetry. At strong coupling we use a holographic model with a pure gauge and mixed gauge-gravitational Chern-Simons term in the action. We discuss the holographic renormalization and show that the Chern-Simons terms do not induce new divergences. Strong and weak coupling results agree precisely. We also point out that the holographic calculation can be done without a singular gauge field configuration on the horizon of the black hole.

1. Introduction

Anomalies are responsible for the breakdown of a classical symmetry due to quantum effects. In vacuum the anomaly appears as the non-conservation of a classically conserved current in a triangle diagram with two additional currents. In four dimension two types of anomalies can be distinguished according to whether only spin one currents appear in the triangle [1, 2] or if also the energy-momentum tensor participates [3, 4]. We will call the first type of anomalies simply chiral anomalies and the second type gravitational anomalies. In four dimension we should actually talk of mixed gauge-gravitational anomalies since triangle diagrams with only energy-momentum insertions are perfectly conserved (see e.g. [5]). In a basis of only left-handed fermions transforming under a symmetry generated by T_A the presence of chiral anomalies is detected by the non-vanishing of $d_{ABC} = \frac{1}{2}\text{Tr}(T_A\{T_B, T_C\})$ whereas the presence of a gravitational anomaly is detected by the non-vanishing of $b_A = \text{Tr}(T_A)$.

Some studies showed that at finite temperature and density, anomalies give rise to new non-dissipative transport phenomena in the hydrodynamics of charged relativistic fluids. In particular magnetic fields in the fluid induce currents via the so-called chiral magnetic effect [6, 7, 8]. Later studies showed that a vortex in a fluid induces also a current parallel to the axial vorticity vector [9, 10]. On the basis of linear response theory, hydrodynamic

transport coefficients can be extracted from the long-wavelength and low-frequency limits of some retarded Green functions. This leads to the so called Kubo formulas. For the chiral magnetic effect the Kubo formula has been derived in [11, 12]. In [13] it was shown that the chiral vortical conductivity for charge and energy transport can be obtained respectively from the retarded Green functions

$$\sigma^{\mathcal{V}} = \lim_{k_c \rightarrow 0} \frac{i}{2k_c} \sum_{a,b} \epsilon_{abc} \langle J^a T^{0b} \rangle|_{\omega=0}, \quad \sigma^{\epsilon, \mathcal{V}} = \lim_{k_c \rightarrow 0} \frac{i}{2k_c} \sum_{a,b} \epsilon_{abc} \langle T^{0a} T^{0b} \rangle|_{\omega=0}, \quad (1)$$

where J^i is the (anomalous) current and T^{ij} is the energy-momentum tensor (see also [14, 15] for details).

In this manuscript we try to understand the effects anomalies have on the transport properties of relativistic fluids, both in the weak and strong coupling regimes, with special emphasis on the gravitational anomaly. Anomalies are very robust features of quantum field theories and do not depend on the details of the interactions. Therefore a non-interacting theory at weak coupling is sufficient for our purpose even without specifying to which gauge theory it corresponds to. By the same way a rather general model that implements the correct anomaly structure in the gauge-gravity setup is enough. Our approach for the latest case will therefore be a “bottom up” approach in which we simply add appropriate Chern-Simons terms that reproduce the relevant anomalies to the Einstein-Maxwell theory in five dimensions with negative cosmological constant.

2. Weak Coupling

In this section we compute the anomalous transport coefficients in a theory of free right-handed fermions Ψ^f transforming under a global symmetry group G generated by matrices $(T_A)^f{}_g$. The chemical potential for the fermion Ψ^f is given by $\mu^f = \sum_A q_A^f \mu_A$, where we write the Cartan generator $H_A = q_A^f \delta^f{}_g$ in terms of its eigenvalues, the charges q_A^f . We define the chemical potential through boundary conditions on the fermion fields around the thermal circle [16]

$$\Psi^f(\tau) = -e^{\beta\mu^f} \Psi^f(\tau - \beta), \quad (2)$$

with $\beta = 1/T$. The currents can be expressed in terms of Dirac fermions as

$$J_A^i = \sum_{f,g=1}^N T_A^g{}_f \bar{\Psi}_g \gamma^i \mathcal{P}_+ \Psi^f, \quad T^{0i} = \frac{i}{2} \sum_{f=1}^N \bar{\Psi}_f (\gamma^0 \partial^i + \gamma^i \partial^0) \mathcal{P}_+ \Psi^f, \quad (3)$$

where we used the chiral projector $\mathcal{P}_{\pm} = \frac{1}{2}(1 \pm \gamma_5)$. The fermion propagator is

$$S(q)^f{}_g = \frac{\delta^f{}_g}{2} \sum_{t=\pm} \Delta_t(i\tilde{\omega}^f, \vec{q}) \mathcal{P}_+ \gamma_\mu \hat{q}_t^\mu, \quad \Delta_t(i\tilde{\omega}^f, q) = \frac{1}{i\tilde{\omega}^f - tE_q}, \quad (4)$$

with $i\tilde{\omega}^f = i\tilde{\omega}_n + \mu^f$, $\hat{q}_t^\mu = (1, t\hat{q})$, $\hat{q} = \frac{\vec{q}}{E_q}$, $E_q = |\vec{q}|$ and $\tilde{\omega}_n = \pi T(2n + 1)$ are the fermionic Matsubara frequencies. We can easily include left-handed fermions as well.

2.1. Vortical conductivity

The vortical conductivity is defined from the retarded correlation function of the current $J_A^i(x)$ and the energy momentum tensor or energy current $T^{0j}(x')$, cf. Eq. (3), i.e.

$$G_A^{\mathcal{V}}(x - x') = \frac{1}{2} \epsilon_{ijn} i \theta(t - t') \langle [J_A^i(x), T^{0j}(x')] \rangle. \quad (5)$$

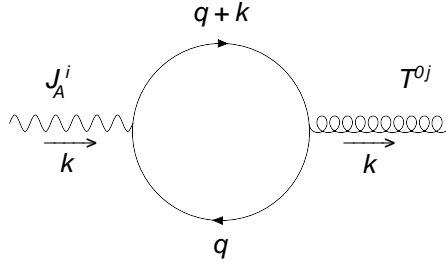


Figure 1. 1 loop diagram contributing to the vortical conductivity Eq. (5).

Going to Fourier space one gets the 1-loop contribution shown in Fig. 1. The result for the zero frequency, zero momentum, vortical conductivity writes [17]

$$(\sigma^\nu)_A = \frac{1}{8\pi^2} d_{ABC} \mu^B \mu^C + \frac{T^2}{24} b_A. \quad (6)$$

The term involving the chemical potentials is induced by the chiral anomaly. More interesting is the term $\sim T^2$ with a coefficient that coincides with the gravitational anomaly coefficient [5]. This means that a non-zero value of this term have to be attributed to the presence of a gravitational anomaly. Left handed fermions contribute in the same way but with a relative minus sign.

2.2. Magnetic conductivity

The magnetic conductivity in the case of a vector and an axial $U(1)$ symmetry was computed at weak coupling in [11]. The corresponding Kubo formula involves the two point function of the current. Following the same method, we get the result for a general symmetry group

$$(\sigma^B)_{AB} = \frac{1}{4\pi^2} d_{ABC} \mu^C. \quad (7)$$

No contribution proportional to the gravitational anomaly coefficient is found in this case.

3. Chemical potentials for anomalous symmetries

Before we go to how these results can be obtained at strong coupling from a gauge-gravity duality we stop for a moment and reflect on the formalism necessary to introduce the chemical potential. Note that we have been quite specific in how we introduced the chemical potentials in the weak coupling calculation in (2). Furthermore this is not the way chemical potentials are commonly discussed in textbooks. Rather than demanding twisted boundary conditions on the thermal circle it is far more common to consider a deformation of the Hamiltonian

$$H \rightarrow H - \mu Q, \quad (8)$$

where Q is the charge in question. We can think of this as arising from the coupling of a (fiducial) gauge field A_μ to the current j^μ of the form $\int d^4x A_\mu j^\mu$ and giving a vacuum expectation value to $A_0 = \mu$. With the fiducial gauge field we have gauge invariance now and we can remove of course the μQ coupling in the Hamiltonian by the gauge transformation $A_0 \rightarrow A_0 + \partial_0 \chi$ with $\chi = -\mu t$. Along the imaginary time direction $t = -i\tau$ this introduces of course just the twist in the boundary conditions on the fields in (2). As long as we have honest non-anomalous symmetries under consideration we have therefore two (gauge)-equivalent formalisms of how to introduce

Table 1. Two formalisms to chemical potential.

Formalism	Hamiltonian	Boundary conditions
(A)	$H - \mu Q$	$\Psi(\tau) = -\Psi(\tau - \beta)$
(B)	H	$\Psi(\tau) = -e^{\beta\mu}\Psi(\tau - \beta)$

the chemical potential summarized in table 1 [18]. One convenient point of view on formalism (B) is the following. In a real time Keldysh-Schwinger setup we demand some initial conditions at initial (real) time $t = t_i$. These initial conditions are given by the boundary conditions in (B). From then on we do the (real) time development with the microscopic Hamiltonian H . This seems an especially suited approach to situations where the charge in question is not conserved by the real time dynamics. In the case of an anomalous symmetry we can start at $t = t_i$ with a state of certain charge but this charge does indeed decay over (real) time due to non-perturbative processes (instantons) or at finite temperature due to thermal sphaleron processes [19]. These processes are however suppressed at large N and so can not be seen easily in the gauge-gravity correspondence. Taking this as excuse we simply ignore them, but keep them in the back of our head as motivation for favoring formalism (B) in the case of an anomalous symmetry, to which we come right now.

Let us assume now that Q is an anomalous charge and start with our favoured formalism (B). We ask what happens if we do now the gauge transformation that would bring us to formalism (A). Since the symmetry is anomalous this means that the action transforms as

$$S[A + \partial\chi] = S[A] + \int d^4x \chi \epsilon^{\mu\nu\rho\lambda} \left(C_1 F_{\mu\nu} F_{\rho\lambda} + C_2 R^\alpha{}_{\beta\mu\nu} R^\beta{}_{\alpha\rho\lambda} \right), \quad (9)$$

with the anomaly coefficients C_1 and C_2 depending on the chiral fermion content. It follows that formalisms (A) and (B) are physically inequivalent now, because of the anomaly. However, we would like to still come as close as possible to the formalism of (A) but in a form that is physically equivalent to the formalism (B). To achieve this we proceed by introducing a non-dynamical axion field $\Theta(x)$ and the vertex

$$S_\Theta[A, \Theta] = \int d^4x \Theta \epsilon^{\mu\nu\rho\lambda} \left(C_1 F_{\mu\nu} F_{\rho\lambda} + C_2 R^\alpha{}_{\beta\mu\nu} R^\beta{}_{\alpha\rho\lambda} \right). \quad (10)$$

If we demand now that the “axion” transforms as $\Theta \rightarrow \Theta - \chi$ under gauge transformations we see that the action

$$S_{tot}[A, \Theta] = S[A] + S_\Theta[A, \Theta] \quad (11)$$

is gauge invariant. Note that this does not mean that the theory is not anomalous now. We introduce it solely for the purpose to make clear how the action has to be modified such that two field configurations related by a gauge transformation are physically equivalent. In other words Θ is a coupling and not a field. The gauge field configuration that corresponds to formalism (B) is simply $A_0 = 0$. A gauge transformation with $\chi = \mu t$ on the gauge invariant action S_{tot} makes clear that a physically equivalent theory is obtained by choosing the field configuration $A_0 = \mu$ and the coupling $\Theta = -\mu t$. If we define the current through the variation of the action with respect to the gauge field we get an additional contribution from S_Θ ,

$$j_\Theta^\mu = 4C_1 \epsilon^{\mu\nu\rho\lambda} \partial_\nu \Theta F_{\rho\lambda}, \quad (12)$$

and evaluating this for $\Theta = -\mu t$ we get the spatial current

$$j_{\Theta}^m = 4C_1\mu B_m, \quad (13)$$

(note that $\epsilon^{0ijk} = -\epsilon_{ijk}$ in Minkowski space). This is *not* the chiral magnetic effect! This is only the contribution to the current that comes from the new coupling that we are forced to introduce by going to formalism (A) from (B) in a (gauge)-equivalent way. The chiral magnetic and vortical effect are on the contrary non-trivial results of dynamical one-loop calculations as shown in the previous section. What is the Hamiltonian now based on the modified formalism (A)? We have to take of course the new coupling generated by the non-zero Θ . The Hamiltonian now is therefore¹

$$H - \mu \left(Q + 4 \int d^3x (C_1 \epsilon^{0ijk} A_i \partial_j A_k + C_2 K^0) \right), \quad (14)$$

where K^0 is the zero component of the gravitational Chern-Simons current $K^\mu = \epsilon^{\mu\nu\rho\lambda} \Gamma_{\beta\nu}^\alpha \left(\partial_\rho \Gamma_{\alpha\lambda}^\beta + \frac{2}{3} \Gamma_{\rho\sigma}^\beta \Gamma_{\alpha\lambda}^\sigma \right)$, fulfilling $\partial_\mu K^\mu = \frac{1}{4} \epsilon^{\mu\nu\rho\lambda} R^\alpha{}_{\beta\mu\nu} R^\beta{}_{\alpha\rho\lambda}$.

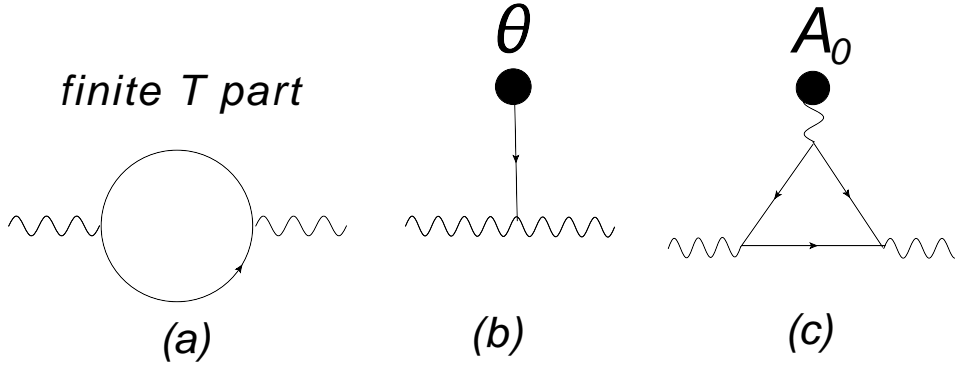


Figure 2. The three different contributions to the current current correlator relevant for the calculation of the chiral magnetic effect. (a) is the (UV-finite) finite temperature part, (b) is the part stemming from the (non-dynamical) axion vertex and (c) stems from the vacuum (T=0) triangle graph coupled to the background gauge field.

Notice that in Formalism (A) we really have three contributions now to the current. One is formally a tree level contribution, namely the current (13), but there are also two different one-loop contributions. The first one comes from the UV-finite part of a finite temperature two point function. This is the part that is typically considered in the weak coupling Kubo formula calculations in [11, 17]. But in formalism (A) we also have a (formally UV-divergent) contribution from the vacuum triangle diagram that couples to the gauge-field background! This has been first pointed out and emphasized in [21]. Graphically in formalism (A) we have to sum the graphs (a), (b) and (c) of figure 2. With $\Theta = -\mu t$ and $A_0 = \mu$ the contributions (b) and (c) cancel each other.

We could also consider the contribution of the axion vertex to the energy-momentum tensor in the case when a gravitational anomaly is present. Since the Riemann tensor is however of second order in derivatives it is clear that the correspondent contribution to the energy current,

¹ This form of the Hamiltonian (without the gravitational part) is the one advocated in [20].

i.e. the T^{0i} components of the energy-momentum tensor, will be of third order in derivatives². In contrast the finite temperature one-loop graphs in Figs. 1 and 2 are first order in derivatives and contribute therefore to ordinary first order hydrodynamics!

Our point of view is that the true anomalous transport effects are captured by the finite temperature graphs (a). These are also the only ones that contribute in formalism (B). We will apply these considerations now to the holographic calculation.

4. Strong Coupling

We present in this section the computation of the anomalous transport coefficients at strong coupling within a holographic model in five dimensions including terms which conveniently mimics the chiral and gauge-gravitational anomalies.

4.1. Holographic Model

Given an outward pointing normal vector $n^A \propto g^{AB} \frac{\partial r}{\partial x^B}$ to the holographic boundary of an asymptotically AdS space with unit norm $n_A n^A = 1$, the induced metric takes the form $h_{AB} = g_{AB} - n_A n_B$. The action of our model is defined by

$$S = \frac{1}{16\pi G} \int d^5x \sqrt{-g} \left[R + 2\Lambda - \frac{1}{4} F_{MN} F^{MN} \right] \quad (15)$$

$$+ \epsilon^{MNPQR} A_M \left(\frac{\kappa}{3} F_{NP} F_{QR} + \lambda R^A{}_{BNP} R^B{}_{AQR} \right) \Big] + S_{GH} + S_{CSK}, \quad (16)$$

$$S_{GH} = \frac{1}{8\pi G} \int_{\partial} d^4x \sqrt{-h} K, \quad (17)$$

$$S_{CSK} = -\frac{1}{2\pi G} \int_{\partial} d^4x \sqrt{-h} \lambda n_M \epsilon^{MNPQR} A_N K_{PL} D_Q K_R^L, \quad (18)$$

where S_{GH} is the usual Gibbons-Hawking boundary term and $D_A = h_A^B \nabla_B$ is the covariant derivative on the four dimensional boundary. The second boundary term S_{CSK} is needed if we want the model to reproduce the gravitational anomaly at general hypersurface. The most important fact about this action is the presence of two Chern-Simons terms. The one proportional to the parameter κ is a pure gauge field CS term and the one proportional to λ a mixed gauge-gravitational CS term. Both of them are diffeomorphism invariant, and they do depend however explicitly on the gauge connection A_M . The bulk equations of motion are

$$G_{MN} - \Lambda g_{MN} = \frac{1}{2} F_{ML} F_N{}^L - \frac{1}{8} F^2 g_{MN} + 2\lambda \epsilon_{LPQR(M} \nabla_B (F^{PL} R^B{}_{N})^{QR}), \quad (19)$$

$$\nabla_N F^{NM} = -\epsilon^{MNPQR} \left(\kappa F_{NP} F_{QR} + \lambda R^A{}_{BNP} R^B{}_{AQR} \right), \quad (20)$$

and they are gauge and diffeomorphism covariant. In Appendix A we discuss the holographic renormalization of the model within the Hamiltonian approach. This leads to the following counterterm of the action [14]

$$S_{ct} = -\frac{(d-1)}{8\pi G} \int_{\partial} d^4x \sqrt{-\gamma} \left[1 + \frac{1}{(d-2)} P - \frac{1}{4(d-1)} \left(P_j^i P_i^j - P^2 - \frac{1}{4} \hat{F}_{(0)ij} \hat{F}_{(0)}^{ij} \right) \log e^{-2r} \right], \quad (21)$$

where $P = \frac{\hat{R}}{2(d-1)}$ and $P_j^i = \frac{1}{(d-2)} [\hat{R}_j^i - P \delta_j^i]$. As a remarkable fact there is no contribution in the counterterm coming from the gauge-gravitational Chern-Simons term. This means that this term does not induce new divergences, and so the renormalization is not modified by it.

² This has recently been made explicit in [22].

4.2. Evaluation of Transport Coefficients

The AdS/CFT dictionary tells us how to compute the retarded propagators [23, 24]. Since we are interested in the linear response limit, we split the metric and gauge field into a background part and a linear perturbation,

$$g_{MN} = g_{MN}^{(0)} + \epsilon h_{MN}, \quad A_M = A_M^{(0)} + \epsilon a_M. \quad (22)$$

Inserting these fluctuations-background fields in the action and expanding up to second order in ϵ one can read the second order action which is needed to get the desired propagators [25]. If we construct a vector Φ^I with the components of a_M and h_{MN} and Fourier transforming it, then it is possible to write the complete second order action on-shell as a boundary term

$$\delta S_{ren}^{(2)} = \int \frac{d^d k}{(2\pi)^d} \left\{ \Phi_{-k}^I \mathcal{A}_{IJ} \Phi_k'^J + \Phi_{-k}^I \mathcal{B}_{IJ} \Phi_k^J \right\} \Big|_{r \rightarrow \infty}. \quad (23)$$

From (23) it is possible to compute the holographic response functions by applying the prescription of [23, 24, 25, 26]. For a coupled system the holographic computation of the correlators consists in finding a maximal set of linearly independent solutions that satisfy infalling boundary conditions on the horizon and that source a single operator at the AdS boundary. To do so we construct a matrix of solutions $F^I{}_J(k, r)$ such that each of its columns corresponds to one of the independent solutions and normalize it to the unit matrix at the boundary. Finally using this decomposition one obtains the matrix of retarded Green functions

$$G_{IJ}(k) = -2 \lim_{r \rightarrow \infty} \left(\mathcal{A}_{IM} (F^M{}_J(k, r))' + \mathcal{B}_{IJ} \right). \quad (24)$$

The system of equations (19)-(20) admit the following exact background AdS Reissner-Nordström black-brane solution

$$ds^2 = \frac{\bar{r}^2}{L^2} \left(-f(\bar{r}) dt^2 + d\bar{x}^2 \right) + \frac{L^2}{\bar{r}^2 f(\bar{r})} d\bar{r}^2, \quad f(\bar{r}) = 1 - \frac{ML^2}{\bar{r}^4} + \frac{Q^2 L^2}{\bar{r}^6}, \quad (25)$$

$$A^{(0)} = \phi(\bar{r}) dt = \left(\alpha - \frac{\mu \bar{r}_H^2}{\bar{r}^2} \right) dt, \quad (26)$$

where the horizon of the black hole is located at $\bar{r} = \bar{r}_H$. We would like to draw attention to the boundary value of the gauge field α . We will make two choices, the choice (A) where $\alpha = \mu$ and the choice (B) where $\alpha = 0$. Note that with choice (A) the gauge field vanished at the horizon! At this point the reader will not be too surprised by our claim that the choice (A) corresponds to the formalism (A) of the previous section and choice (B) to the formalism (B). We know now however, that if we want (A) to be physically equivalent to (B) we also need to take into account the pure boundary action $S_\Theta[A, \Theta]$. In the holographic setup this is a *pure* boundary term, Θ does not extend into the bulk and is therefore not dynamical. If we do not include this coupling it is known that the choices (A) and (B) give different results. In fact it has caused quite some confusion that the choice (A) without taking into account the Θ -coupling gives a vanishing result for the chiral magnetic effect as shown in [8]. The alternative approach (B) causes however some uneasiness because now the gauge field necessarily is singular at the horizon (though not the field strength), but it gives the correct result for the chiral magnetic effect with no need of including the pure boundary action $S_\Theta[A, \Theta]$, as it was done in [21]. The nice feature of the modified approach (A) in holography is that we can now happily make the gauge field vanish at the horizon and still obtain the correct result for the anomalous conductivities on the boundary ☺

The parameters M , Q and Hawking temperature of the RN black hole write

$$M = \frac{\bar{r}_H^4}{L^2} + \frac{Q^2}{\bar{r}_H^2}, \quad Q = \frac{\mu \bar{r}_H^2}{\sqrt{3}}, \quad T = \frac{\bar{r}_H^2}{4\pi L^2} f'(\bar{r}_H) = \frac{(2\bar{r}_H^2 M - 3Q^2)}{2\pi \bar{r}_H^5}. \quad (27)$$

We just consider momentum fluctuations of the fields transverse to the perturbations of the momentum, i.e. we focus on the shear sector. Then one arrives at a system of four second order differential equations. The relevant physical boundary conditions on fields are: $h_t^\alpha(0) = \tilde{H}^\alpha$, $B_\alpha(0) = \tilde{B}_\alpha$; where the ‘tilde’ parameters are the sources of the boundary operators. The second condition compatible with the ingoing one at the horizon is regularity for the gauge field and vanishing for the metric fluctuation [13].

After solving the system of differential equations perturbatively (see [14] for details), one can go back to the formula (24) and compute the corresponding holographic Green functions. Finally, by using the Kubo formulas (1) one recovers the conductivities

$$\sigma^{\mathcal{B}} = -\frac{\sqrt{3} Q \kappa}{2\pi G \bar{r}_H^2} = \frac{\mu}{4\pi^2}, \quad (28)$$

$$\sigma^{\mathcal{V}} = \sigma^{\epsilon, \mathcal{B}} = -\frac{3 Q^2 \kappa}{4\pi G \bar{r}_H^4} - \frac{2\lambda\pi T^2}{G} = \frac{\mu^2}{8\pi^2} + \frac{T^2}{24}, \quad (29)$$

$$\sigma^{\epsilon, \mathcal{V}} = -\frac{\sqrt{3} Q^3 \kappa}{2\pi G \bar{r}_H^6} - \frac{4\pi\sqrt{3} Q T^2 \lambda}{G \bar{r}_H^2} = \frac{\mu^3}{12\pi^2} + \frac{\mu T^2}{12}, \quad (30)$$

where we have included for completeness the chiral magnetic conductivity $\sigma^{\mathcal{B}}$ and the magnetic conductivity for energy current $\sigma^{\epsilon, \mathcal{B}}$.

In a hydrodynamic framework we can summarize our findings in the constitutive relations

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P\eta^{\mu\nu} + Q^\mu u^\nu + Q^\nu u^\mu, \quad (31)$$

$$J^\mu = nu^\mu + N^\mu, \quad (32)$$

with the first order in derivative terms

$$N^\mu = \sigma^{\mathcal{B}} \mathcal{B}^\mu + \sigma^{\mathcal{V}} \Omega^\mu, \quad (33)$$

$$Q^\mu = \sigma^{\epsilon, \mathcal{B}} \mathcal{B}^\mu + \sigma^{\epsilon, \mathcal{V}} \Omega^\mu. \quad (34)$$

For simplicity of the expressions we have dropped here the usual dissipative terms related to shear and bulk viscosities or electric conductivity. The equilibrium quantities ϵ, P, n are energy density, pressure and charge density and we defined the (covariant) magnetic field $\mathcal{B}^\mu = \epsilon^{\mu\nu\rho\lambda} u_\nu \partial_\rho A_\lambda$ and vorticity vector $\Omega^\mu = \epsilon^{\mu\nu\rho\lambda} u_\nu \partial_\rho u_\lambda$.

According to the discussion above, these values for the coefficients can be obtained, either by using the action of the holographic model and setting the deformation parameter α to zero (choice B), or considering the total action $S_{tot}[A, \Theta]$, Eq. (11), and setting $\alpha = \mu$ and $\Theta = -\mu t$ (choice A). In the latest case the axion term (10) induces a contribution in matrices \mathcal{A}_{IJ} and \mathcal{B}_{IJ} , see Eq. (23). The expression for $\sigma^{\mathcal{B}}$ is in perfect agreement with the literature and the one for $\sigma^{\mathcal{V}}$ shows the extra T^2 term already predicted in [17] and shown in Section 2. In fact the numerical coefficients coincide precisely with the ones obtained in weak coupling when specifying Eqs. (6) and (7) to the $U(1)_R$ group. This we take as a strong hint that the anomalous conductivities are indeed completely determined by the anomalies and are not renormalized beyond one loop.

5. Discussion and conclusion

In the presence of external sources for the energy momentum tensor and the currents, the anomaly is responsible for a non conservation law of the latter. This is conveniently expressed through [5]

$$\nabla_\mu J_A^\mu = \epsilon^{\mu\nu\rho\lambda} \left(\frac{d_{ABC}}{32\pi^2} F_{\mu\nu}^B F_{\rho\lambda}^C + \frac{b_A}{768\pi^2} R^\alpha{}_{\beta\mu\nu} R^\beta{}_{\alpha\rho\lambda} \right). \quad (35)$$

The axial and mixed gauge-gravitational anomaly coefficients are defined respectively by

$$d_{ABC} = \frac{1}{2}[\text{tr}(T_A\{T_B, T_C\})_R - \text{tr}(T_A\{T_B, T_C\})_L], \quad (36)$$

$$b_A = \text{tr}(T_A)_R - \text{tr}(T_A)_L, \quad (37)$$

where the subscripts R, L stand for the contributions of right-handed and left-handed fermions.

We have computed the magnetic and vortical conductivity of a relativistic fluid at weak coupling and we find contributions that are proportional to the anomaly coefficients (36) and (37). Non-zero values of these coefficients are a necessary and sufficient condition for the presence of anomalies [5]. Therefore the non-vanishing values of the transport coefficients (6) and (7) have to be attributed to the presence of chiral and gravitational anomalies.

In order to perform the analysis at strong coupling via AdS/CFT methods, we have defined a holographic bottom up model that implements both anomalies in four dimensional field theory via gauge and mixed gauge-gravitational Chern-Simons terms. We have computed the anomalous magnetic and vortical conductivities from a charged black hole background and have found a non-vanishing vortical conductivity proportional to $\sim T^2$. These terms are characteristic for the contribution of the gravitational anomaly and they even appear in an uncharged fluid. The T^2 behavior had appeared already previously in neutrino physics [27, 28]. In [29] similar terms in the vortical conductivities have been argued for without any relation to the gravitational anomaly. However so far the effects of gravitational anomalies have not been taken into account in the purely hydrodynamic treatments, and therefore the T^2 terms appear simply as undetermined integration constants. The numerical values of the anomalous conductivities computed at strong coupling are in perfect agreement with weak coupling calculations, and this suggests the existence of a non-renormalization theorem including the contributions from the gravitational anomaly.

So far we have computed the transport coefficients, and in particular their gravitational anomaly contributions, via Kubo formulas. It would be interesting to calculate directly the constitutive relations of the hydrodynamics of anomalous currents via the fluid/gravity correspondence within the model used in the present paper [9, 30, 31]. This study is currently in progress [32].

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Appendix A. Holographic Renormalization

In order to have a deeper understanding of the holographic model, we will compute the renormalized action within the Hamiltonian approach, see e.g. [33, 34]. Without loss of generality we choose a gauge in which $A_r = 0$ and the bulk metric writes

$$ds^2 = dr^2 + \gamma_{ij}dx^i dx^j. \quad (A.1)$$

Latin letters denote four dimensional (boundary) indices. The non vanishing Christoffel symbols are $\Gamma_{ij}^r = -K_{ij} = -\frac{1}{2}\dot{\gamma}_{ij}$ and $\Gamma_{jr}^i = K_j^i$, where dot denotes differentiation respect r . Then one can compute the off shell action in terms of transverse components of tensors. It is useful to divide the action up in three terms:

$$S^0 = \frac{1}{16\pi G} \int d^5x \sqrt{-\gamma} \left[\hat{R} + 2\Lambda + K^2 - K_{ij}K^{ij} - \frac{1}{2}E_i E^i - \frac{1}{4}\hat{F}_{ij}\hat{F}^{ij} \right], \quad (A.2)$$

$$S_{CS}^1 = -\frac{\kappa}{12\pi G} \int d^5x \sqrt{-\gamma} \epsilon^{ijkl} A_i E_j \hat{F}_{kl}, \quad (\text{A.3})$$

$$S_{CS}^2 = -\frac{8\lambda}{16\pi G} \int d^5x \sqrt{-\gamma} \epsilon^{ijkl} \left[A_i \hat{R}^n{}_{mkl} D_n K_j^m + E_i K_{jm} D_k K_l^m + \frac{1}{2} \hat{F}_{ik} K_{jm} \dot{K}_l^m \right]. \quad (\text{A.4})$$

The first one is the usual gravitational bulk and gauge terms with the usual Gibbons-Hawking term. Of particular concern is the last term in S_{CS}^2 which contains explicitly the normal derivative of the extrinsic curvature \dot{K}_{ij} . Then the field equations will be generically of third order in r -derivatives. Having applications to holography in mind we can however impose the boundary condition that the metric has an asymptotically AdS expansion of the form

$$\gamma_{ij} = e^{2r} \left(g_{ij}^{(0)} + e^{-2r} g_{ij}^{(2)} + e^{-4r} (g_{ij}^{(4)} + 2r \tilde{g}_{ij}^{(4)}) + \dots \right), \quad r \rightarrow \infty. \quad (\text{A.5})$$

Using this expansion one can see that the last term in the action does not contribute in the limit $r \rightarrow \infty$. Therefore the boundary action depends only on the boundary metric γ_{ij} but not on the derivative $\dot{\gamma}_{ij}$.

The renormalization procedure follows from an expansion of the four dimensional quantities in eigenfunctions of the dilatation operator

$$\delta_D = 2 \int d^4x \gamma_{ij} \frac{\delta}{\delta \gamma_{ij}}. \quad (\text{A.6})$$

For the extrinsic curvature and gauge fields, this expansion reads respectively

$$K_j^i = K_{(0)}^i{}_j + K_{(2)}^i{}_j + K_{(4)}^i{}_j + \tilde{K}_{(4)}^i{}_j \log e^{-2r} + \dots, \quad (\text{A.7})$$

$$A_i = A_{(0)i} + A_{(2)i} + \tilde{A}_{(2)i} \log e^{-2r} + \dots, \quad (\text{A.8})$$

where the subindexes denote the corresponding eigenvalue, except for the cases $\delta_D K_{(4)}^i{}_j = -4K_{(4)}^i{}_j - 2\tilde{K}_{(4)}^i{}_j$ and $\delta_D A_{(2)i} = -2A_{(2)i} - 2\tilde{A}_{(2)i}$. Given the above expansion of the fields one has to solve the equations of motion in its Codazzi form, order by order in a recursive way. The derivative on r can be computed by using

$$\frac{d}{dr} = \int d^4x \dot{\gamma}_{km} \frac{\delta}{\delta \gamma_{km}} = 2 \int d^4x K_m^l \gamma_{lk} \frac{\delta}{\delta \gamma_{km}}. \quad (\text{A.9})$$

By inserting in this equation the expansion of K_j^i given by Eq. (A.7), one gets $d/dr \simeq \delta_D$ at the lowest order. Taking into account this, the computation of $K_{(0)}^i{}_j$ is trivial, i.e.

$$K_{(0)ij} = \frac{1}{2} \dot{\gamma}_{ij}|_{(0)} = \frac{1}{2} \delta_D \gamma_{ij} = \gamma_{ij}, \quad K_{(0)} = d. \quad (\text{A.10})$$

By following the procedure and using the Codazzi form of equations of motion up to fourth order [14], one gets

$$K_{(2)} := P = \frac{\hat{R}}{2(d-1)}, \quad K_{(2)}^i{}_j := P_j^i = \frac{1}{(d-2)} \left[\hat{R}_j^i - P \delta_j^i \right], \quad (\text{A.11})$$

$$K_{(4)} = \frac{1}{2(d-1)} \left[P_j^i P_i^j - P^2 - \frac{1}{4} \hat{F}_{(0)ij} \hat{F}_{(0)}^{ij} \right], \quad \tilde{K}_{(4)} = 0. \quad (\text{A.12})$$

In order to compute the counterterm for the on-shell action, besides the equations of motion an additional equation is needed. Following Ref. [34], one can introduce a covariant variable θ and write the on-shell action as

$$S_{on-shell} = \frac{1}{8\pi G} \int_{\partial} d^4x \sqrt{-\gamma} (K - \theta). \quad (\text{A.13})$$

The variable θ admits also an expansion in eigenfunctions of δ_D of the form $\theta = \theta_{(0)} + \theta_{(2)} + \theta_{(4)} + \tilde{\theta}_{(4)} \log e^{-2r} + \dots$. Using the corresponding Codazzi equation for θ [14], and the same procedure as above, one gets the result

$$\theta_{(0)} = 1, \quad \theta_{(2)} = \frac{P}{(2-d)}. \quad (\text{A.14})$$

$$\tilde{\theta}_{(4)} = \frac{1}{4} \left[P_j^i P_i^j - P^2 - \frac{1}{4} \hat{F}_{(0)ij} \hat{F}_{(0)}^{ij} + \frac{1}{3} D_i \left(D^i P - D^j P_j^i \right) \right]. \quad (\text{A.15})$$

Finally the counterterm of the action can be read out from Eq. (A.13) by using K and θ computed up to fourth order. The result is

$$S_{ct} = -\frac{(d-1)}{8\pi G} \int_{\partial} d^4x \sqrt{-\gamma} \left[1 + \frac{1}{(d-2)} P - \frac{1}{4(d-1)} \left(P_j^i P_i^j - P^2 - \frac{1}{4} \hat{F}_{(0)ij} \hat{F}_{(0)}^{ij} \right) \log e^{-2r} \right]. \quad (\text{A.16})$$

We have explicitly checked that the λ dependence starts contributing at sixth order. So the gauge-gravitational Chern-Simons term does not induce new divergences, and there is no contribution in the counterterm coming from it.

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